Gravitational waves from hyperbolic encounters

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The emission of gravitational waves from a system of massive objects interacting on hyperbolic orbits is studied in the quadrupole approximation. Analytic expressions are derived for the gravitational radiation luminosity, the total energy output and the gravitational radiation amplitude. An estimation of the expected number of events towards different targets (i.e. globular clusters and the center of the Galaxy) is also given. In particular, for a dense stellar cluster at the galactic center, a rate up to one event per year is obtained.

Keywords: gravitational radiation; quadrupole approximation; theory of orbits.

General Relativity predicts that a system of interacting massive objects emits gravitational waves which propagate in the vacuum with the speed of light. A lot of studies have been devoted to the description of gravitational radiation emitted by systems of two interacting stars where amplitude, power and all physical quantities of such a radiation strictly depend on the configuration and the dynamics of the system. The seminal papers by Peters and Mathews [1, 2], which investigate the gravitational wave emission by a binary system of stars (on circular or elliptic orbits) in the quadrupole approximation, have been extended in several directions (see e.g. [3] and references therein) in which the problem is studied by both analytical and numerical approaches.

On the other hand, depending on their approaching energy, stars may interact also on unbound orbits (parabolic or hyperbolic) and, in this case, one expects that gravitational waves are emitted with a peculiar signature of a "burst" wave-form having a maximum in correspondence of the peri-astron distance. The problem of *Bremsstrahlung*-like gravitational wave emission has been studied in detail by Kovacs and Thorne [4] by considering stars interacting on unbound orbits.

Here, we face this problem discussing the dynamics of such a phenomenon in the simplest case where stars interact on unbound orbits and the peri-astron distance is much larger than the Schwarzschild radius of the stars. In this case, the quadrupole approximation holds and useful analytic quantities (as the energy emitted by the system per unit time and gravitational wave amplitude) can be derived.

In this letter, after reviewing the main features of hyperbolic encounters between two stars, the emission of gravitational waves is studied in the framework of the quadrupole approximation. Particular attention is devoted to the gravitational radiation luminosity. Then, we derive the expected gravitational wave-form and discuss the detection of such events which could greatly improve the statistics of possible gravitational wave sources.

The study of gravitational wave emission from massive objects interacting on hyperbolic orbits can be started by analyzing the geometry of hyperbolic encounters. Without loss of generality, let us consider a mass M_1 moving in the gravitational potential Φ generated by a second mass M_2 at rest in the center O of the reference frame (see Fig. 1). Let ON be a reference direction in the orbital plane. Then, the position of M_1 (in P) is specified by the vector radius $\mathbf{r}$ and by the polar angle ϕ which the vector radius forms with ON, ϕ being measured in the direction of the star motion. Obviously, both the vector radius and the polar angle depend on time as a consequence of the star motion, i.e. $\mathbf{r} = \mathbf{r}(t)$ and $\phi = \phi(t)$. With this choice, the velocity $\mathbf{v}$ of the mass M_1 can be parameterized as

$$\mathbf{v} = v_r \widehat{r} + v_\phi \widehat{\phi} \, , \tag{1}$$

where the radial and the tangent components of the velocity are, respectively,

$$v_r = \frac{dr}{dt} \qquad v_\phi = r\frac{d\phi}{dt} \, . \tag{2}$$

In this case, the total energy and the angular momentum, per unit mass, read out

$$E = \frac{1}{2}v^2 + \Phi(r) = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}r^2\left(\frac{d\phi}{dt}\right)^2 + \Phi(r) \tag{3}$$

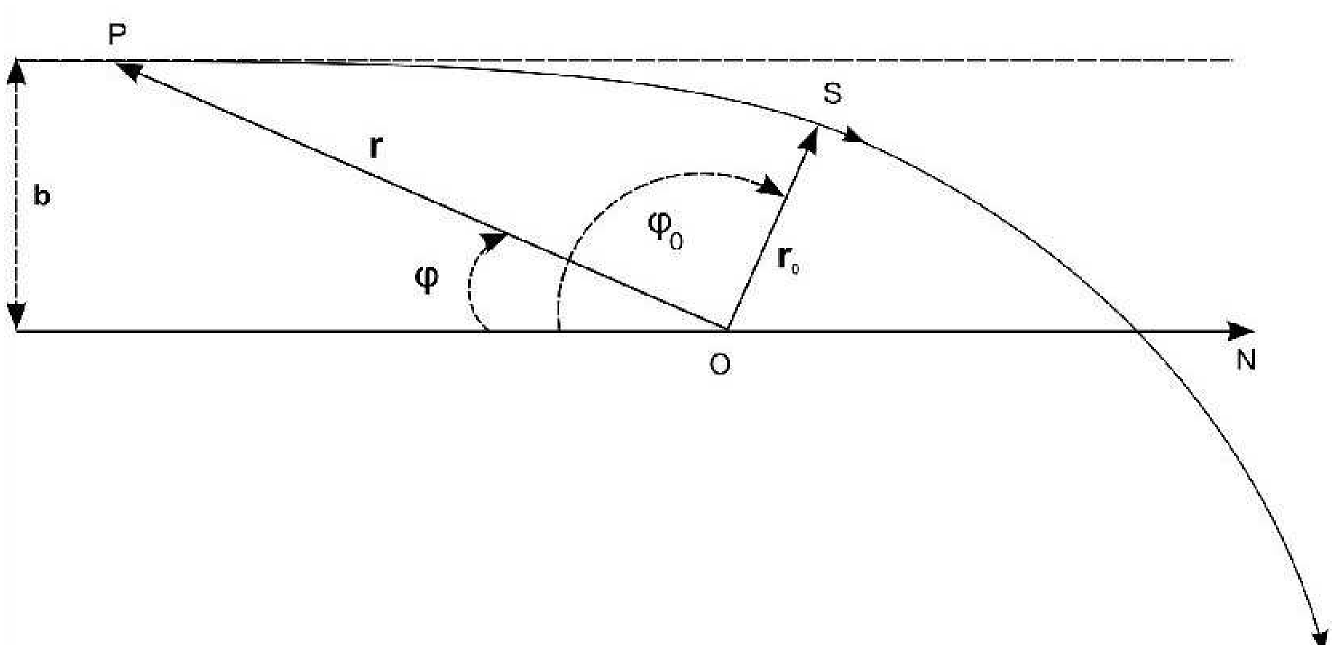

Figure 1: The geometry of the hyperbolic encounter. The mass M_1 (in P) is moving on hyperbolic orbit (continuous line) with focus in O where the mass M_2 lies. The motion of M_1 is described by the vector radius $\mathbf{r}$ and the polar angle ϕ. The vector radius $\mathbf{r}_0$ (corresponding to the polar angle ϕ_0) represents the peri-astron distance, i.e. the distance of the closest approach between the two interacting stars.

and

$$L = r^2 \frac{d\phi}{dt} \ , \tag{4}$$

respectively. At this point, it is useful to adopt the variable $u = 1/r$ so that eq. (3) can be rewritten as

$$\left(\frac{du}{d\phi}\right)^2 + u^2 + \frac{2\Phi(r)}{L^2} = \frac{2E}{L^2} = \text{const.} \ , \tag{5}$$

which, differentiated with respect to u, gives

$$\frac{d^2u}{d\phi^2} + u = -\frac{1}{L^2}\frac{d\Phi(u)}{du} \ . \tag{6}$$

The force acting on the unit mass is $F(r) = -\frac{GM_2}{r^2} = -GM_2u^2$ which can be expressed as the gradient of the gravitational potential Φ. Hence, the last equation can be recast in the form [5]

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{L^2u^2}F\left(\frac{1}{u}\right) \ , \tag{7}$$

and thus

$$\frac{d^2u}{d\phi^2} + u = \frac{GM_2}{L^2} \ , \tag{8}$$

whose general solution, giving the parameterized trajectory followed by the unit mass particle, is

$$u(\phi) = C\cos(\phi - \phi_0) + \frac{GM_2}{L^2} \ . \tag{9}$$

Here, C is a constant depending on the initial conditions and ϕ_0 is the polar angle corresponding to the peri-astron distance, i.e. the distance of closest approach between the two interacting particles. Note that after the interaction, the particle is deflected by the angle $\phi_d = 2\phi_0$.

For a system of two objects (of mass M_1 and M_2) interacting on hyperbolic orbits, the two-body problem can be reduced to the problem of a reduced mass particle $\mu = M_1 M_2/(M_1 + M_2)$ moving in the gravitational field generated by a total fictitious mass $M = M_1 + M_2$. Let us assume that M_2 is at rest while M_1 is moving with initial (at infinite distance) velocity v_0 with impact parameter b (see Fig. 1).

Considering what previously stated, it is then clear that the trajectory followed by the reduced mass particle is

$$u(\phi) = C\cos(\phi - \phi_0) + \frac{G(M_1 + M_2)}{L^2} \ . \tag{10}$$

Differentiating the previous relation with respect to the time t and turning back to the variable r, one obtains

$$\frac{dr}{dt} = CL\sin(\phi - \phi_0) \ , \tag{11}$$

by which it possible to determine the value of the constant C through the initial conditions of the motion, i.e.

$$C = \frac{v_0}{L\sin(\phi_0)} \ , \tag{12}$$

which can be written, being $L = bv_0$, as

$$C = \frac{1}{b\sin\phi_0} \ . \tag{13}$$

Alternatively, the initial conditions of the motion can be used directly in eq. (10). In this case, one finds

$$C = -\frac{G(M_1 + M_2)}{b^2\cos\phi_0} \ , \tag{14}$$

which compared with eq. (13) gives the result

$$\tan\phi_0 = -\frac{bv_0^2}{G(M_1 + M_2)} \ . \tag{15}$$

Clearly, eq.(13) is meaningless for $\phi_0 = 90^o$, which means that, for this critical value, there is no interaction between the stars which are at an infinite distance each other. In other words, stars are very far when ϕ approaches to 90^o.

Finally, the equation of the orbit followed by the reduced mass particle turns out to be [6]

$$r = \frac{b\sin\phi_0}{\cos(\phi - \phi_0) - \cos\phi_0} \ , \tag{16}$$

which allows to determine the modulus of the radius vector $\mathbf{r}$ as a function of the polar angle ϕ once the initial conditions are known. With this set of equations at hand, we can estimate both the gravitational wave luminosity of the system and the gravitational radiation wave-form in the quadrupole approximation. We are adopting such an approximation since we are considering "binary" systems whose distance is larger than the "capture" distance. In other words, we are considering situations and initial conditions where pointlike approximation of stars holds and the system remains unbounded. In such cases, quadrupole approximation works and results are reasonable [7].

The Einstein field equations give a description of how the curvature of space-time, at any event, is related to the energy-momentum distribution at that event. In the weak field approximation, it is found that systems of massive moving objects produce gravitational waves which propagate in the vacuum with the speed of light. It can be shown that the energy emitted per unit time, in the form of gravitational radiation (after integrating on all the gravitational wave polarization states), is [8]

$$\frac{dE}{dt} = -\frac{G\left\langle D_{ij}^{(3)} D^{(3)ij} \right\rangle}{45c^5} \tag{17}$$

where the dot represents the differentiation with respect to time, the symbol $\langle\rangle$ indicates the scalar product and the quadrupole mass tensor D_{ij} is defined as

$$D_{ij} = \sum_a m_a(3x_a^i x_a^j - \delta_{ij} r_a^2) \ , \tag{18}$$

r_a being the modulus of the vector radius of the $a-th$ particle.

It is then possible to estimate the amount of energy emitted in the form of gravitational waves from a system of massive objects interacting on hyperbolic orbits. In this case, the components of the quadrupole mass tensor are

$$
\begin{aligned}
D_{11} &= \mu r^2(3\cos^2\phi - 1) \ , \\
D_{22} &= \mu r^2(3\sin^2\phi - 1) \ , \\
D_{12} &= D_{21} = 3\mu r^2 \cos\phi \sin\phi \ , \\
D_{33} &= -\mu r^2,
\end{aligned}
\tag{19}
$$

which can be differentiated with respect to time as required in eq. (17). In doing this, we can use some useful relations derived above, in particular eqs. (4), (11), (15) and (16). It is straightforward to show that

$$
D_{ij}^{(3)} D^{(3)ij} = \frac{32 L^6 \mu^2}{b^8} f(\phi, \phi_0) \ , \tag{20}
$$

where $f(\phi, \phi_0)$ is given by

$$
\begin{aligned}
f(\phi, \phi_0) = {} & \sin^4(\phi_0 - \phi/2)\sin^4(\phi/2) \ \tan^{-2}\phi_0 \ \sin^{-6}\phi_0 \\
& \times [150 + 72\cos 2\phi_0 + 66\cos 2(\phi_0 - \phi) \\
& -144\cos(2\phi_0 - \phi) - 144\cos\phi] \ .
\end{aligned}
\tag{21}
$$

Hence, the energy emitted by the system per unit time is

$$
\frac{dE}{dt} = -\frac{32 G L^6 \mu^2}{45 c^5 b^8} f(\phi, \phi_0) \ , \tag{22}
$$

or, equivalently,

$$
\frac{dE}{dt} = -\frac{32 G v_0^6 \mu^2}{45 c^5 b^2} f(\phi, \phi_0) \ , \tag{23}
$$

which, for $M_1 = M_2$, can be rewritten as

$$
\frac{dE}{dt} = -\frac{4 r_s v_0^6 m}{45 c^3 b^2} f(\phi, \phi_0) \ , \tag{24}
$$

r_s being the Schwarzschild radius of the mass.

The total energy emitted in the form of gravitational radiation during the interaction is given by

$$
\Delta E = \int_0^\infty |\frac{dE}{dt}| dt \ . \tag{25}
$$

Since eq. (4) holds, we can adopt the angle ϕ as a convenient integration variable. In this case, the energy emitted for $\phi_1 < \phi < \phi_2$ is

$$
\Delta E(\phi_1, \phi_2) = \frac{4 r_s m v_0^5}{45 c^3 b} \int_{\phi_1}^{\phi_2} \frac{\sin^2\phi_0 \ f(\phi, \phi_0)}{[\cos(\phi - \phi_0) - \cos\phi_0]^2} \ d\phi \ , \tag{26}
$$

and the total energy can be determined from the previous relation in the limits $\phi_1 \to 0$ and $\phi_2 \to 2\phi_0$. Thus, one has

$$
\Delta E = \frac{v_0^5 r_s m}{c^3} F(b, v_0) \ , \tag{27}
$$

where $F(b, v_0)$ only depends on the initial conditions and it is given by

$$
\begin{aligned}
F(b, v_0) = {} & [720 b \tan^2\phi_0 \sin^4\phi_0]^{-1} \times (2628\phi_0 \\
& +2328\phi_0 \cos 2\phi_0 + 144\phi_0 \cos 4\phi_0 \\
& -1948 \sin 2\phi_0 - 301 \sin 4\phi_0) \ .
\end{aligned}
\tag{28}
$$

In other words, the gravitational radiation luminosity strictly depends on the configuration and kinematics of the binary system and it improves at short b and high v_0.

Direct signatures of gravitational radiation are its amplitude and its wave-form. In other words, the identification of a gravitational radiation signal is strictly related to the accurate selection of the shape of wave-forms by interferometers or any possible detection tool. Such an achievement could give information on the nature of the gravitational wave source, on the propagating medium, and even, in general, on the gravitational theory producing such a radiation [9, 10]. It is well known that the amplitude of the gravitational waves can be evaluated as

$$h^{jk}(t,R) = \frac{2G}{Rc^4}\ddot{D}^{jk} \ , \tag{29}$$

R being the distance between the source and the observer and $\{j,k\} = 1,2$.

Considering our binary system and the single components of eq.(29), it is straightforward to show that

$$\begin{array}{l}
h^{11} = \frac{2G}{Rc^4}\frac{\csc^2(\phi_0)\mu v_0^2}{4}\left[13\cos\phi - 12\cos 2\phi + 3\cos 3\phi\right.\\
-2\cos(\phi - 2\phi_0) + 3\cos(3\phi - 2\phi_0) - 12\cos(2\phi_0)\\
\left.-6\cos 2(\phi_0 - \phi) - 6\cos 2(\phi + \phi_0) + 15\cos(\phi + 2\phi_0) + 4\right] \ ,\\
\\
h^{22} = \frac{2G}{Rc^4}\frac{\csc^2(\phi_0)\mu v_0^2}{4}\left[-17\cos\phi + 12\cos 2\phi - 3\cos 3\phi\right.\\
-2\cos(\phi - 2\phi_0) - 3\cos(3\phi - 2\phi_0) + 12\cos 2\phi_0\\
\left.+6\cos 2(\phi_0 - \phi) + 6\cos 2(\phi + \phi_0) - 15\cos(\phi + 2\phi_0) + 4\right] \ ,\\
\\
h^{12} = h^{21} = \frac{2G}{Rc^4}3\mu v_0^2 \csc^2\phi_0 \sin^2\phi/2\left[2\sin\phi - \sin 2\phi\right.\\
\left.-\sin 2\phi_0 + \sin 2(\phi_0 - \phi) + 2\sin(\phi + 2\phi_0)\right] \ ,
\end{array} \tag{30}$$

so that the expected strain amplitude $h \simeq (h_{11}^2 + h_{22}^2 + 2h_{12}^2)^{1/2}$ turns out to be

$$\begin{aligned}
h = \frac{2G}{Rc^4}\mu v_0^2 \csc^2\phi_0 &\left\{2\left[59\cos 2(\phi_0 - \phi) - \cos\phi(54\cos(2\phi_0) + 101)\right]\cos^2\phi_0\right.\\
&-9\cos(3\phi - 4\phi_0) - 9\cos(3\phi - 2\phi_0) + 95\cos 2\phi_0 + 9\cos 4\phi_0 -\\
&\left.\sin\phi\left[101\sin 2\phi_0 + 27\sin 4\phi_0\right] + 106\right\}^{1/2} \ ,
\end{aligned} \tag{31}$$

which, as before, strictly depends on the initial conditions of the stellar encounter. A remark is in order at this point. A monochromatic gravitational wave has, at most, two independent degrees of freedom while eq. (30) seems to show three independent parameters associated to the amplitude of the corresponding wave. This is not true since, as usual in the TT gauge, we have h_+ and $h_\times$ being $h_+ = h_{11} + h_{22}$ and $h_\times = h_{12} + h_{21}$. For details, see [7].

As an example, the amplitude of gravitational wave is sketched in Fig.2 for a stellar encounter close to the Galactic Center. The adopted initial parameters are typical of a close hyperbolic impact and are assumed to be $b = 1$ AU and $v_0 = 200$ K$\mathrm{m}s^{-1}$, respectively. Here, we have fixed $M_1 = M_2 = 1.4M_\odot$.

In the following, we give an estimate of the rate of stellar encounters on hyperbolic orbits in some interesting astrophysical conditions as a typical globular cluster or towards the bulge of our Galaxy.

Let us consider a generic star cluster with mass density profile $\rho(r)$ composed, for the sake of simplicity, by N_* stars of equal mass m moving with relative velocities $v_{\rm rel}(r)$. Then, the event rate Γ, that is the number of interacting stars on hyperbolic orbits per unit time, is given by

$$\Gamma = \int_{R_0}^{R}\left[\frac{\rho(r)}{m}\right]^2 \sigma(r) v_{\rm rel}(r) 4\pi r^2 dr \ . \tag{32}$$

Here, $\rho(r)$ is the cluster density profile (which we assume to follow a Plummer model [5]), R its radius and $\sigma(r) \simeq \pi[b_1(r)^2 - b_2(r)^2]$ the typical cross section of the hyperbolic encounter at distance r from the cluster center (see [11] for details). In evaluating the rate, we are considering only those hyperbolic encounters producing gravitational waves, for example, in the LISA range, i.e. between 10^{-4} and 10^{-2} Hz (see e.g. [12]). The integral on the right hand of the previous equation can be approximately solved and put in the form

$$\Gamma \simeq 5.5 \times 10^{-10}\left(\frac{M}{10^5 \mathrm{M}_\odot}\right)^2\left(\frac{v}{10\mathrm{kms}^{-1}}\right)\left(\frac{\sigma}{UA^2}\right)\left(\frac{10\mathrm{pc}}{R}\right)^3 \mathrm{yrs}^{-1}. \tag{33}$$

For a typical globular cluster (GC) around the Galactic Center, the expected event rate is of the order of 2×10^{-9} yrs^{-1} which may be increased at least by a factor $\simeq 100$ if one considers the number of GCs in the Galaxy. If the

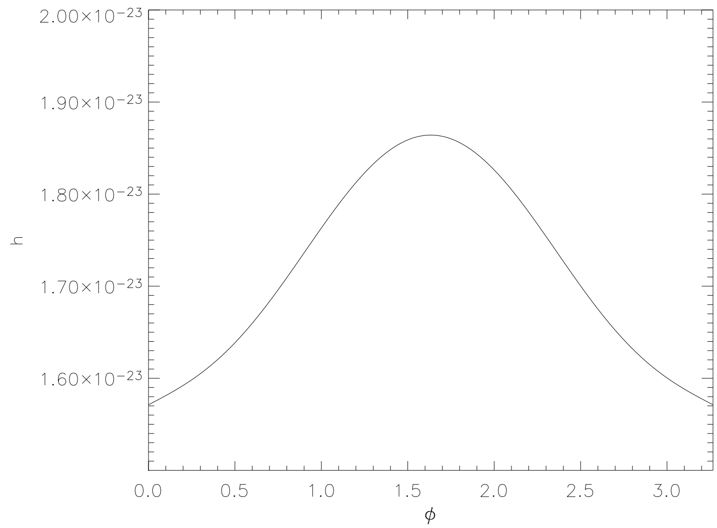

Figure 2: The gravitational wave-form is sketched as a function of the polar angle ϕ for some values of both the impact parameter and velocity. In particular, we have fixed $M_1 = M_2 = 1.4 M_{\odot}$. M_2 is considered at rest while M_1 is moving with initial velocity $v_0 = 200$ Kms^{-1} and an impact parameter $b = 1$ AU. The distance of the source has been assumed to be $r = 8$ kpc. As expected, the wave-form has a maximum in correspondence of the peri-astron distance.

stellar cluster at the galactic center is taken into account and assuming $M \simeq 3 \times 10^6$ M$_{\odot}$, $v \simeq 150$ km s^{-1} and $R \simeq$ 10 pc, one expects to have $\simeq 10^{-5}$ open orbit encounters per year. If a cluster with total mass $M \simeq 10^6$ M$_{\odot}$, $v \simeq 150$ km s^{-1} and $R \simeq 0.1$ pc is considered, an event rate number of the order of unity per year is obtained. These values could be realistically achieved by data coming from the forthcoming space interferometer LISA.

In this letter, the gravitational wave emission on hyperbolic stellar encounters has been analyzed. In particular, we have taken into account the expected luminosity and the strain amplitude of gravitational radiation produced in a tight hyperbolic impact where two massive objects of $1.4 M_{\odot}$ closely interact at an impact distance of $1AU$. Due to the high probability of such encounters inside rich stellar fields (e.g. globular clusters, bulges of galaxies and so on), the presented approach could highly contribute to enlarge the classes of gravitational wave sources (in particular, of dynamical phenomena capable of producing gravitational waves). In particular, a detailed theory of stellar orbits could improve the statistics of possible gravitational wave sources.

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